Oscillation Criteria for First Order Linear Delay Difference Equations

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Abstract. Oscillation criteria for all solutions of the first order delay difference equation of the form

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, ...,$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer are established especially in the case that the well-known oscillation conditions

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}$$

are not satisfied. Our results essentially improve known results in the literature.

Key words: Oscillation, nonoscillation, delay difference equation.

AMS Subject Classification (2000): 39A 11.

1. INTRODUCTION

In the last few decades the oscillation theory of delay differential equations has been extensively developed. The oscillation theory of discrete analogues of delay differential equations has also attracted growing attention in the recent few years. The reader is referred to [1-16, 18-32] and the references cited therein. In particular, the problem of establishing sufficient conditions for the oscillation of all solutions of the delay difference equation

$$\Delta x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, ..., \tag{1.1}$$

where $\{p_n\}$ is a sequence of nonnegative real numbers, k is a positive integer, and Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$, has been the subject of many recent investigations. See, for example, [2-9, 12-16, 18-27, 29-32] and the references cited therein. Strong interest in Eq. (1.1) is motivated by the fact that it represents a discrete analogue of the delay differential equation (see [18] and the references cited therein)

$$x'(t) + p(t)x(t-\tau) = 0, \quad p(t) \geq 0, \quad \tau > 0. \tag{1.2}$$

By a solution of (1.1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies (1.1) for $n \geq 0$. A solution $\{x_n\}$ of (1.1) is said to be *oscillatory* if the terms x_n of the solution are not eventually positive or eventually negative. Otherwise the solution is called *nonoscillatory*.

For convenience, we will assume that inequalities about values of sequences are satisfied eventually for all large n.

In this paper, our main purpose is to derive new oscillation conditions for all solutions to Eq. (1.1), especially in the case that the known oscillation conditions (see below)

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}$$

are not satisfied.

2. OSCILLATION CRITERIA FOR EQ. (1.1)

In 1981, Domshlak [3] was the first who studied this problem in the case where $k = 1$. Then, in 1989 Erbe and Zhang [9] established the following oscillation criteria for Eq. (1.1).

Theorem 2.1.([9]) *Assume that*

$$\beta := \liminf_{n\to\infty} p_n > 0 \quad \textit{and} \quad \limsup_{n\to\infty} p_n > 1 - \beta \tag{C_1}$$

Then all solutions of Eq. (1.1) *oscillate.*

Theorem 2.2.([9]) *Assume that*

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}} \tag{C_2}$$

Then all solutions of Eq. (1.1) *oscillate.*

Theorem 2.3.([9]) *Assume that*

$$A := \limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 \tag{C_3}$$

Then all solutions of (1.1) *oscillate.*

In the same year 1989 Ladas, Philos and Sficas [13] proved the following theorem.

Theorem 2.4.([13]) *Assume that*

$$\liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}. \tag{C_4}$$

Then all solutions of Eq. (1.1) *oscillate.*

Therefore they improved the condition (C_2) by replacing the p_n of (C_2) by the arithmetic mean of the terms $p_{n-k}, \ldots, p_{n-1}$ in (C_4).

Concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (C_2) and (C_4) it should be empasized that, as it is shown in [9], if

$$\sup p_n < \frac{k^k}{(k+1)^{k+1}} \tag{N_1}$$

then Eq. (1.1) has a nonoscillatory solution.

In 1990, Ladas [12] conjectured that Eq. (1.1) has a nonoscillatory solution if

$$\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \le \frac{k^k}{(k+1)^{k+1}}$$

holds eventually. However this conjecture is not true and a counterexample was given in 1994 by Yu, Zhang and Wang [30].

It is interesting to establish sufficient conditions for the oscillation of all solutions of (1.1) when (C_3) and (C_4) are not satisfied. (For Eq. (1.2), this question has been investigated by many authors, see, for example, [17] and the references cited therein).

In 1993, Yu, Zhang and Qian [29] and Lalli and Zhang [14], trying to improve (C_3), established the following (false) sufficient oscillation conditions for Eq. (1.1)

$$0 < \alpha := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i \le \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad A > 1 - \frac{\alpha^2}{4} \tag{F_1}$$

and

$$\sum_{i=n-k}^{n} p_i \geq d > 0 \text{ for large } n \text{ and } A > 1 - \frac{d^4}{8}\left(1 - \frac{d^3}{4} + \sqrt{1 - \frac{d^3}{2}}\right)^{-1} \qquad (F_2)$$

respectively.

Unfortunately, the above conditions (F_1) and (F_2) are not correct. This is due to the fact that they are based on the following (false) discrete version of Koplatadze-Chanturija Lemma [12]. (See [6] and [2]).

Lemma A (False). *Assume that $\{x_n\}$ is an eventually positive solution of Eq. (1.1) and that*

$$\sum_{i=n-k}^{n} p_i \geq M > 0 \quad \textit{for large } n. \tag{1.3}$$

Then

$$x_n > \frac{M^2}{4} x_{n-k} \quad \textit{for large } n.$$

As one can see, the erroneous proof of Lemma A is based on the following (false) statement. (See [6] and [2]).

Statement A (False). *If (1.3) holds, then for any large N, there exists a positive integer n such that $n - k \leq N \leq n$ and*

$$\sum_{i=n-k}^{N} p_i \geq \frac{M}{2}, \quad \sum_{i=N}^{n} p_i \geq \frac{M}{2}.$$

It is obvious that all the oscillation results which have made use of the above Lemma A or Statement A are not correct. For details on this problem see the paper by Cheng and Zhang [2].

Here it should be pointed out that the following statement (see [13], [20]) is correct and it should not be confused with the Statement A.

Statement 2.1. ([13], [20]) *If*

$$\sum_{i=n-k}^{n-1} p_i \geq M > 0 \quad \textit{for large } n, \tag{1.4}$$

then for any large n, there exists a positive integer n^* with $n - k \leq n^* \leq n$ such that

$$\sum_{i=n-k}^{n^*} p_i \geq \frac{M}{2}, \quad \sum_{i=n^*}^{n} p_i \geq \frac{M}{2}.$$

In 1995, Stavroulakis [20], based on this correct Statement 2.1, proved the following theorem.

Theorem 2.5.([20]) *Assume that*

$$0 < \alpha \leq \left(\frac{k}{k+1}\right)^{k+1}$$

and

$$\limsup_{n\to\infty} p_n > 1 - \frac{\alpha^2}{4}. \tag{C_5}$$

Then all solutions of Eq. (1.1) *oscillate.*

In 1998, Domshlak [5], studied the oscillation of all solutions and the existence of nonoscillatory solution of Eq. (1.1) with r -periodic positive coefficients $\{p_n\}, p_{n+r} = p_n$. It is very important that in the following cases where $\{r = k\}, \{r = k+1\}, \{r = 2\}, \{k = 1, r = 3\}$ and $\{k = 1, r = 4\}$ the results obtained are stated in terms of necessary and sufficient conditions and it is very easy to check them.

Following this historical (and chronological) review we also mention that in the case where

$$\frac{1}{k}\sum_{i=n-k}^{n-1} p_i \geq \frac{k^k}{(k+1)^{k+1}} \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{k}\sum_{i=n-k}^{n-1} p_i = \frac{k^k}{(k+1)^{k+1}},$$

the oscillation of (1.1) has been studied in 1994 by Domshlak [4] and in 1998 by Tang [21] (see also Tang and Yu [23]). In a case when p_n is asymptotically close to one of the periodic critical states, unimprovable results about oscillation preperties of the equation

$$x_{n+1} - x_n + p_n x_{n-1} = 0$$

were obtained by Domshlak in 1999 [7] and in 2000 [8].

In 1999, Domshlak [6] and in 2000 Cheng and Zhang [2] established the following lemmas, respectively, which may be looked upon as (exact) discrete versions of Koplatadze-Chanturia Lemma.

Lemma 2.1. ([6]) *Assume that* $\{x_n\}$ *is an eventually positive solution of Eq. (1.1) and that*

$$\sum_{i=n-k}^{n-1} p_i \geq M > 0 \quad \textit{for large } n. \tag{1.4}$$

Then

$$x_n > \frac{M^2}{4} x_{n-k} \quad \textit{for large } n. \tag{1.5}$$

Lemma 2.2. ([2]) *Assume that $\{x_n\}$ is an eventually positive solution of Eq. (1.1) and that*

$$\sum_{i=n-k}^{n-1} p_i \geq M > 0 \quad \textit{for large } n. \tag{1.4}$$

Then

$$x_n > M^k x_{n-k} \quad \textit{for large } n. \tag{1.6}$$

In 2004, Stavroulakis [21], based on the above two lemmas, established the following theorem.

Theorem 2.6. *Assume that*

$$0 < \alpha \leq \left(\frac{k}{k+1}\right)^{k+1}.$$

Then either one of the conditions

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \frac{\alpha^2}{4} \tag{C_6}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \alpha^k \tag{C_7}$$

implies that all solutions of Eq. (1.1) *oscillate.*

Remark 2.1. From the above theorem it is now clear that

$$0 < \alpha := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i \leq \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \frac{\alpha^2}{4}$$

is the correct oscillation condition by which the (false) condition (F_1) should be replaced.

In the following lemma (cf. [6]) we establish an improvement for the upper bound for $\frac{x_{n-k}}{x_n}$. Then using this (improved) upper bound we derive a condition which essentially improves the conditions (C_6) and (C_7).

Lemma 2.3. *Assume that $\{x_n\}$ is an eventually positive solution of Eq. (1.1) and that*

$$\sum_{i=n-k}^{n-1} p_i \geq M > 0 \quad \textit{for large } n. \tag{1.4}$$

Then

$$x_n > \frac{M^2}{2(2-M)} x_{n-k} \quad \textit{for large } n. \tag{1.7}$$

Proof. Since $\{x_n\}$ is an eventually positive solution of Eq. (1.1), then eventually

$$\Delta x_n = x_{n+1} - x_n \leq -p_n x_{n-k} \leq 0,$$

and so $\{x_n\}$ is an eventually nonincreasing sequence of positive numbers. For all n consider the following two possible cases: (i) $p_n \geq \frac{M}{2}$, and (ii) $p_n < \frac{M}{2}$.

In the case (i), from Eq. (1.1), it is clear that

$$x_n = x_{n+1} + p_n x_{n-k} \geq x_{n+1} + \frac{M}{2} x_{n-k}.$$

Also, summing up Eq. (1.1) from $n-k$ to $n-1$, and using the fact that the sequence $\{x_n\}$ is nonincreasing, we have

$$x_{n-k} - x_n = \sum_{i=n-k}^{n-1} p_i x_{i-k} \geq \left(\sum_{i=n-k}^{n-1} p_i\right) x_{n-k-1} \geq \frac{M}{2} x_{n-k}$$

or

$$x_{n-k} \geq x_n + \frac{M}{2} x_{n-k}.$$

From the last two inequalities we obtain

$$x_n \geq x_{n+1} + \frac{M}{2}\left(x_n + \frac{M}{2} x_{n-k}\right)$$

which implies that

$$x_n > \frac{M^2}{2(2-M)} x_{n-k}.$$

In the case (ii), there exists n^*, $n+1 \leq n^* \leq n+k$, such that $\sum_{i=n}^{n^*-1} p_i < \frac{M}{2}$ and $\sum_{i=n}^{n^*} p_i \geq \frac{M}{2}$.

Therefore

$$\sum_{i=n^*-k}^{n-1} p_i = \sum_{i-n^*-k}^{n^*-1} p_i - \sum_{i=n}^{n^*-1} p_i \geq M - \frac{M}{2} = \frac{M}{2}.$$

Moreover summing up Eq. (1.1) first from n to n^* and then from $n^* - k$ to $n-1$, and using the fact that the sequence $\{x_n\}$ is nonincreasing, we have

$$x_n - x_{n^*+1} = \sum_{i=n}^{n^*} p_i x_{i-k} \geq \left(\sum_{i=n}^{n^*} p_i \right) x_{n^*-k} \geq \frac{M}{2} x_{n^*-k}$$

that is,

$$x_n \geq x_{n^*+1} + \frac{M}{2} x_{n^*-k}$$

and

$$x_{n^*-k} - x_n = \sum_{i=n^*-k}^{n-1} p_i x_{i-k} \geq \left(\sum_{i=n^*-k}^{n-1} p_i \right) x_{n-k-1} \geq \frac{M}{2} x_{n-k}$$

that is,

$$x_{n^*-k} \geq x_n + \frac{M}{2} x_{n-k}.$$

Combining the last two inequalities, we obtain

$$x_n \geq x_{n^*+1} + \frac{M}{2} \left(x_n + \frac{M}{2} x_{n-k} \right)$$

that is,

$$x_n > \frac{M^2}{2(2-M)} x_{n-k}.$$

The proof is complete.

Theorem 2.7. *Assume that*

$$0 < \alpha \leq \left(\frac{k}{k+1} \right)^{k+1}.$$

and

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \frac{\alpha^2}{2(2-\alpha)}. \qquad (C_8)$$

Then all solutions of Eq. (1.1) *oscillate.*

Proof. Assume, for the sake of contadiction, that $\{x_n\}$ is an eventually positive solution of Eq. (1.1). Then eventually

$$\Delta x_n = x_{n+1} - x_n \leq -p_n x_{n-k} \leq 0,$$

and so $\{x_n\}$ is an eventually nonincreasing sequence of positive numbers. Summing up Eq. (1.1) from $n-k$ to $n-1$, we have

$$x_n - x_{n-k} + \sum_{i=n-k}^{n-1} p_i x_{i-k} = 0,$$

and, because $\{x_n\}$ is eventually nonincreasing, it follows that for all sufficiently large n

$$x_n - x_{n-k} + \left(\sum_{i=n-k}^{n-1} p_i \right) x_{n-k} \leq 0,$$

or

$$x_{n-k} \left(\sum_{i=n-k}^{n-1} p_i + \frac{x_n}{x_{n-k}} - 1 \right) \leq 0.$$

Now, using Lemma 2.3, for all sufficiently large n, we have

$$x_{n-k} \left(\sum_{i=n-k}^{n-1} p_i + \frac{\alpha^2}{2(2-\alpha)} - 1 \right) \leq 0$$

which, in view of (C_8), leads to a contradiction. The proof is complete.

Remark 2.2. Observe the following:

(i) When $\alpha \longrightarrow 0$, then it is clear that the conditions (C_6), (C_7) and (C_8) reduce to

$$\mathbf{A} := \limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1,$$

which obviously implies (C_3).

(ii) It always holds

$$\frac{\alpha^2}{2(2-\alpha)} > \frac{\alpha^2}{4},$$

sinca $\alpha > 0$ and therefore condition (C_6) always implies (C_8).

(iii) When $k = 1, 2$

$$\frac{\alpha^2}{2(2-\alpha)} < \alpha^k,$$

(since, from the above mentioned conditions, it makes sense to investigate the case when $\alpha \le \left(\frac{k}{k+1}\right)^{k+1}$) and therefore condition (C_8) implies (C_7).

(iv) When $k = 3$,

$$\frac{\alpha^2}{2(2-\alpha)} > \alpha^3 \text{ if } 0 < \alpha < 1 - \frac{\sqrt{2}}{2}$$

while

$$\frac{\alpha^2}{2(2-\alpha)} < \alpha^3 \text{ if } 1 - \frac{\sqrt{2}}{2} < \alpha \le \left(\frac{3}{4}\right)^4.$$

So in this case the conditions (C_8) and (C_7) are independent.

(v) When $k \ge 4$

$$\frac{\alpha^2}{2(2-\alpha)} > \alpha^k,$$

and therefore condition (C_7) implies (C_8).

(vi) When $k \ge 10$ condition (C_8) may hold but condition (C_3) may not hold.

(vii) When k is large then $\alpha \longrightarrow \frac{1}{e}$ and ih this case both conditions (C_6) and (C_7) imply (C_8). For illustartive purposes, we give the values of the lower bound of **A** under these conditions when $k = 100$ $(\alpha \simeq 0.366)$:

$$\begin{aligned} (C_7) &: 0.999999 \\ (C_6) &: 0.966511 \\ (C_8) &: 0.959009 \end{aligned}$$

We see that our condition (C_8) essentially improve the conditions (C_6) and (C_7).

We illustrate these by the following examples.

Example 2.1. Consider the equation

$$x_{n+1} - x_n + p_n x_{n-3} = 0, \quad n = 0, 1, 2, ...,$$

where

$$p_{2n} = \frac{1}{10}, \quad p_{2n+1} = \frac{1}{10} + \frac{6731}{10000}\sin^2\frac{n\pi}{2}, \qquad n = 0, 1, 2, ...,$$

Here $k = 3$ and it is easy to see that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{3}{10} < \left(\frac{3}{4}\right)^4 \simeq 0.3164$$

and

$$\limsup_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{3}{10} + \frac{6731}{10000} > 1 - \alpha^3 = 0.973.$$

Thus condition (C_7) is satisfied and therefore all solutions oscillate. Observe, however, that condition (C_8) is not satisfied.

If, on the other hand, in the above equation

$$p_{2n} = \frac{8}{100}, \quad p_{2n+1} = \frac{8}{100} + \frac{744}{1000}\sin^2\frac{n\pi}{2}, \qquad n = 0, 1, 2, ...,$$

then it is easy to see that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} < \left(\frac{3}{4}\right)^4 \simeq 0.3164$$

and

$$\limsup_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} + \frac{744}{1000} > 1 - \frac{\alpha^2}{2(2-\alpha)} \simeq 0.9836.$$

In this case condition (C_8) is satisfied and therefore all solutions oscillate. Observe, however, that condition (C_7) is not satisfied.

Example 2.2. Consider the equation

$$x_{n+1} - x_n + p_n x_{n-10} = 0, \quad n = 0, 1, 2, ...,$$

where

$$p_{11n+1} = ... = p_{11n+10} = \frac{35}{1000}, \quad p_{11n+11} = \frac{35}{1000} + \frac{613}{1000}, \quad n = 0, 1, 2, ...$$

Here $k = 10$ and it is easy to see that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-10}^{n-1} p_i = \frac{35}{100} < \left(\frac{10}{11}\right)^{11} \simeq 0.3504$$

and

$$\limsup_{n\to\infty} \sum_{i=n-10}^{n-1} p_i = \frac{35}{100} + \frac{613}{1000} = 0.963 > 1 - \frac{\alpha^2}{2(2-\alpha)} \simeq 0.9628.$$

We see that condition (C_8) is satisfied and therefore all solutions oscillate. Observe, however, that

$$0.963 < 1 - \frac{\alpha^2}{4} \simeq 0.9693,$$

$$0.963 < 1 - \alpha^{10} \simeq 0.9999,$$

and

$$A = \limsup_{n\to\infty} \sum_{i=n-10}^{n} p_i = \frac{35}{100} + \frac{963}{1000} = 0.998 < 1.$$

Therefore none of the conditions (C_6), (C_7) and (C_3) is satisfied.

REFERENCES

1. R.P. Agarwal and P.J.Y. Wong, Advanced Topics in Difference Equations, Kluwer Academic Publishers, 1997.

2. S. S. Cheng and G. Zhang, "Virus" in several discrete oscillation theorems, *Applied Math. Letters*, **13** (2000), 9-13.

3. Y. Domshlak, Discrete version of Sturmian Comparison Theorem for non-symmetric equations, *Doklady Azerb.Acad.Sci.* **37** (1981), 12-15 (in Russian).

4. Y. Domshlak, Sturmian comparison method in oscillation study for discrete difference equations, I, *Differential and Integral Equations*, **7** (1994), 571-582.

5. Y. Domshlak, Delay-difference equations with periodic coefficients: sharp results in oscillation theory, *Math. Inequal. Appl.*, **1** (1998), 403-422.

6. Y. Domshlak, What should be a discrete version of the Chanturia-Koplatadze Lemma? *Funct. Differ. Equ.*, **6** (1999), 299-304.

7. Y. Domshlak, Riccati Difference Equations with almost periodic coefficients in the critical state, *Dynamic Systems Appl.*, **8** (1999), 389-399.

8. Y. Domshlak, The Riccati Difference Equations near "extremal" critical states, *J. Difference Equations Appl.*, **6** (2000), 387-416.

9. L. Erbe and B.G. Zhang, Oscillation of discrete analogues of delay equations, *Differential and Integral Equations*, **2** (1989), 300-309.

10. I. Györi and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.

11. J. Jaroš and I.P. Stavroulakis, Necessary and sufficient conditions for oscillations of difference equations with several delays, *Utilitas Math.*, **45** (1994), 187-195.

12. R.G. Koplatadze and T.A. Chanturija, On the oscillatory and and monotonic solutions of first order differential equations with deviating arguments, *Differentsial'nye Uravneniya,* **18** (1982), 1463-1465.

13. G. Ladas, Recent developments in the oscillation of delay difference equations, In *International Conference on Differential Equations, Stability and Control*, Dekker, New York, 1990.

14. G. Ladas, C. Philos and Y. Sficas, Sharp conditions for the oscillation of delay difference equations, *J. Appl. Math. Simulation*, **2** (1989), 101-112.

15. B. Lalli and B.G. Zhang, Oscillation of difference equations, *Colloquium Math.*, **65** (1993), 25-32.

16. Zhiguo Luo and J.H. Shen, New results for oscillation of delay difference equations, *Comput. Math. Appl.* **41** (2001), 553-561.

17. Zhiguo Luo and J.H. Shen, New oscillation criteria for delay difference equations, *J. Math. Anal. Appl.* **264** (2001), 85-95.

18. Y.G. Sficas and I.P. Stavroulakis, Oscillation criteria for first-order delay differential equations, *Bull. London Math. Soc.* **35** (2003), no.2, 239-246.

19. J.H. Shen and Zhiguo Luo, Some oscillation criteria for difference equations, *Comput. Math. Applic.*, **40** (2000), 713-719.

20. J.H. Shen and I.P. Stavroulakis, Oscillation criteria for delay difference equations, Univ. of Ioannina, T. R. N^0 4, 2000, *Electron. J. Diff. Eqns. Vol.* **2001** (2001), no.10, pp. 1-15.

21. I.P. Stavroulakis, Oscillation of delay difference equations, *Comput. Math. Applic.*, **29** (1995), 83-88.

22. I.P. Stavroulakis, Oscillation Criteria for First Order Delay Difference Equations, *Mediterr. J. Math.*, **1** (2004), 231-240.

23. X.H. Tang, Oscillations of delay difference equations with variable coefficients, (Chinese), *J. Central South Univ. of Technology*, **29** (1998), 287-288.

24. X.H. Tang and S.S. Cheng, An oscillation criterion for linear difference equations with oscillating coefficients, J. *Comput. Appl. Math..*, **132** (2001), 319-329.

25. X.H. Tang and J.S. Yu, Oscillation of delay difference equations, *Comput. Math. Applic.*, **37** (1999), 11-20.

26. X.H. Tang and J.S. Yu, A further result on the oscillation of delay difference equations, *Comput. Math. Applic.*, **38** (1999), 229-237.

27. X.H. Tang and J.S. Yu, Oscillations of delay difference equations in a critical state, *Appl. Math. Letters*, **13** (2000), 9-15.

28. X.H. Tang and J.S. Yu, Oscillation of delay difference equations, *Hokkaido Math. J.* **29** (2000), 213-228.

29. X.H. Tang and J.S. Yu, New oscillation criteria for delay difference equations, *Comput. Math. Applic.*, **42** (2001), 1319-1330.

30. Weiping Yan and Jurang Yan, Comparison and oscillation results for delay difference equations with oscillating coefficients, *Internat. J. Math. & Math. Sci.*, **19** (1996), 171-176.

31. J.S. Yu, B.G. Zhang and X.Z. Qian, Oscillations of delay difference equations with oscillating coefficients, *J. Math. Anal. Appl.*, **177** (1993), 432-444.

32. J.S. Yu, B.G. Zhang and Z.C. Wang, Oscillation of delay difference equations, *Appl. Anal.*, **53** (1994), 117-124.

33. B.G. Zhang and Yong Zhou, The semicycles of solutions of delay difference equations, *Comput. Math. Applic.*, **38** (1999), 31-38.

34. B.G. Zhang and Yong Zhou, Comparison theorems and oscillation criteria for difference equations, *J. Math. Anal. Appl.*, **247** (2000), 397-409.